ON POSSIBLE MIXING RATES FOR SOME STRONG MIXING CONDITIONS FOR N-TUPLEWISE INDEPENDENT RANDOM FIELDS

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Abstract. For a given pair of positive integers d and N with $N \geq 2$, for strictly stationary random fields that are indexed by the d-dimensional integer lattice and satisfy N-tuplewise independence, the dependence coefficients associated with the ρ-, ρ'-, and ρ^*-mixing conditions can decay together at an arbitrary rate. Another, closely related result is also established. In particular, these constructions provide classes of examples pertinent to limit theory for random fields that involve such mixing conditions together with certain types of "extra" assumptions on the marginal and bivariate (or N-variate) distributions.

AMS 2010 Mathematics Subject Classifications: 60G10, 60G60

Key words and phrases: Random field, mixing conditions, mixing rates, N-tuplewise independence

1. Introduction

In certain types of limit theorems for random fields under strong mixing conditions, there are sometimes "extra conditions" involving the marginal and (say) bivariate distributions. For example, in a paper by the author and Tran [7], a central limit theorem for some kernel-type estimators of probability density was proved for strictly stationary random fields that satisfy certain strong mixing conditions as well as certain "extra" assumptions: an absolutely continuous marginal distribution with a continuous density function, and a dependence condition on the bivariate distributions. In a subsequent paper, the author [3] constructed a class of random fields that satisfy all of those conditions (including the "extra" ones); and by altering the choices of parameters in those examples, one can get practically arbitrary "mixing rates" within certain rather loose constraints. Later, Tone [24, Theorem 5.2] proved a functional central limit theorem for empirical processes from strictly stationary ρ'-mixing random fields that satisfy the following "extra" conditions: the marginal distribution is absolutely continuous and (for convenience) supported on the unit interval, and there exists a positive constant C such that the bivariate distributions are absolutely continuous with bivariate density functions that are (at any given point (x, y)) bounded above by C times the product of the marginal densities. (The mixing conditions in the papers [7] and [3] cited above were quite different from ρ'-mixing, and will not be dealt with further in this note. The ρ'-mixing condition itself will be defined below.) The author [5] constructed a class of strictly stationary ρ'-mixing random fields with arbitrary mixing rates. That construction gave further information on other related mixing conditions (more on that below); however, it did not address the question whether such ρ'-mixing examples exist which also satisfy certain types of "extra conditions" (on the marginal and bivariate distributions) such as in the result of Tone [24] cited above. In this note, that question will be answered affirmatively, à la the paper [3] cited above.

Those "extra conditions" in the cited result from [24] are trivially satisfied in the case where the random variables are (say) uniformly distributed on the unit interval and are pairwise independent. In the examples that are constructed in this note, the ρ'-mixing condition will (for appropriate choices of parameters) be satisfied with an arbitrary mixing rate, the marginal distributions will be uniform on the unit interval, and the random variables will be pairwise independent (and satisfy an even stronger "independence" property). The examples given below will also involve three other mixing conditions closely related to ρ'-mixing; motivations for that will be given below.

Before the main results are stated, some notations and definitions need to be given.

Throughout this paper, the setting will be a probability space $(\Omega, \mathcal{F}, P)$ rich enough to accommodate all random variables defined here. For any collection $(Y_i, i \in I)$ of random variables defined on this probability space, the σ-field of events generated by this collection will be denoted $\sigma(Y_i, i \in I)$.

Let $\mathbf{N}$ (resp. $\mathbf{R}$ resp. $\mathbf{Z}$) denote the set of all positive integers (resp. all real numbers resp. all integers).

Suppose d is a positive integer. A given element $k \in \mathbf{Z}^d$ will be represented by

$k := (k_1, k_2, \dots, k_d)$. For any element $k \in \mathbf{Z}^d$, define the Euclidean norm $\|k\| := (k_1^2 + k_2^2 + \dots + k_d^2)^{1/2}$. The origin in $\mathbf{Z}^d$ will be denoted with boldface $\mathbf{0} := (0, 0, \dots, 0)$. For any two nonempty disjoint subsets S and T of $\mathbf{Z}^d$, define the positive number $\text{dist}(S, T) := \min_{s \in S, t \in T} \|s - t\|$.

(Of course if $d = 1$, then $\|k\| = |k|$, $\mathbf{0} = 0$, and $\text{dist}(S, T) = \min_{s \in S, t \in T} |s - t|$.)

Definition 1.1. Suppose d is a positive integer, and $X := (X_k, k \in \mathbf{Z}^d)$ is a (not necessarily stationary) random field on $(\Omega, \mathcal{F}, P)$. Suppose $N \geq 2$ is an integer. The random field X satisfies "N-tuplewise independence" if the following holds: For every choice of N distinct elements $k(1), k(2), \dots, k(N) \in \mathbf{Z}^d$, the random variables $X_{k(1)}, X_{k(2)}, \dots, X_{k(N)}$ are independent.

The notion of N-tuplewise independence — and in particular, pairwise independence (the case $N = 2$) — is of interest in its own right. Limit theory for sequences of pairwise independent random variables has been developed in a consider number of papers, including the ones by Etemadi [12][13] and Etemadi and Lenzhen [14]. The paper [13] cites several references for practical applications of pairwise independence (e.g. in statistics and computer science). Also, some limitations of that theory have been illustrated with, for example, nontrivial pairwise independent counterexamples to the central limit theorem, such as constructions in the papers of Janson [17], of Cuesta and Matrán [9], and of the author and Pruss [6]. The example in the latter paper satisfies N-tuplewise independence, where N is an arbitrary integer ≥ 2 specified beforehand. In the book by McWilliams and Sloane [19], in connection with the construction of error-correcting codes, some methodology involves the construction of "big" N-tuplewise independent random vectors from "small" ones. The paper by Rosenblatt and Slepian [23] dealt with the question of whether (for a given N and n) N-tuplewise independence can hold for nth order Markov chains with certain restrictions on the state space.

Now let us turn our attention to mixing conditions.

Definition 1.2. For any two σ-fields $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{F}$, define the following two measures of dependence:

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(A \cap B) - P(A)P(B)| \quad \text{and} \tag{1.1}$$

$$\rho(\mathcal{A}, \mathcal{B}) := \sup |\text{Corr}(f, g)| \tag{1.2}$$

where the latter supremum is taken over all pairs of square-integrable random variables f and g such that f is $\mathcal{A}$-measurable and g is $\mathcal{B}$-measurable. The quantity $\rho(\mathcal{A}, \mathcal{B})$ is known as the "maximal correlation coefficient" between $\mathcal{A}$ and $\mathcal{B}$. The following inequality is elementary (see e.g. [4, v1, Proposition 3.11(b)]): For any two σ-fields $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{F}$,

$$4\alpha(\mathcal{A}, \mathcal{B}) \leq \rho(\mathcal{A}, \mathcal{B}). \tag{1.3}$$

Definition 1.3. Again suppose d is a positive integer, and $X := (X_k, k \in \mathbf{Z}^d)$ is a (not necessarily stationary) random field on $(\Omega, \mathcal{F}, P)$. (No assumption of "N-tuplewise independence.") For each positive integer n, referring to (1.1) and (1.2), define the following

four "dependence coefficients": First,

$$\alpha(n) = \alpha(X,n) := \sup \alpha(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \quad \text{and} \tag{1.4}$$
$$\rho(n) = \rho(X,n) := \sup \rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \tag{1.5}$$

where in each of (1.4) and (1.5) the supremum is taken over all pairs of sets S and $T \subset \mathbf{Z}^d$ such that for some integer j and some $u \in \{1, 2, \dots, d\}$, one has that $S = \{k \in \mathbf{Z}^d : k_u \le j\}$ and $T = \{k \in \mathbf{Z}^d : k_u \ge j + n\}$. Next,

$$\rho'(n) = \rho'(X,n) := \sup \rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \tag{1.6}$$

where this supremum is taken over all pairs of sets S and $T \subset \mathbf{Z}^d$ such that for some $u \in \{1, 2, \dots, d\}$ and some pair of nonempty, disjoint sets G and $H \subset \mathbf{Z}$ such that $\text{dist}(G, H) \ge n$, one has that $S = \{k \in \mathbf{Z}^d : k_u \in G\}$ and $T = \{k \in \mathbf{Z}^d : k_u \in H\}$. (Note that the sets G and H may be "interlaced," with each one containing elements between ones in the other set.) Finally,

$$\rho^*(n) = \rho^*(X,n) := \sup \rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \tag{1.7}$$

where this supremum is taken over all pairs of nonempty, disjoint sets S and $T \subset \mathbf{Z}^d$ such that $\text{dist}(S, T) \ge n$.

Obviously, for each $n \in \mathbf{N}$, by (1.3),

$$0 \le 4\alpha(n) \le \rho(n) \le \rho'(n) \le \rho^*(n) \le 1. \tag{1.8}$$

Also obviously, the sequences $(\alpha(n), n \in \mathbf{N})$, $(\rho(n), n \in \mathbf{N})$, $(\rho'(n), n \in \mathbf{N})$, and $(\rho^*(n), n \in \mathbf{N})$ are each nonincreasing as n increases. The random field X is said to be
"strongly mixing," or "α-mixing," if $\alpha(n) \to 0$ as $n \to \infty$,
"ρ-mixing" if $\rho(n) \to 0$ as $n \to \infty$,
"ρ'-mixing" if $\rho'(n) \to 0$ as $n \to \infty$, and
"ρ^*-mixing" if $\rho^*(n) \to 0$ as $n \to \infty$.

Our focus will be on strictly stationary random fields. There is a vast literature on limit theory for strictly stationary random fields satisfying the mixing conditions above and closely related ones. See e.g. [4, v3], [7], [15], [16], [21], [22], [24], and [25], to name just a few references. As in the well known paper of Kesten and O'Brien [18] (involving random sequences — the case $d = 1$), there has been an ongoing interest in the question of what "mixing rates" are possible for the above mixing conditions and related ones, especially under strict stationarity. The papers [1], [3], and [5] deal with that question for various mixing conditions for strictly stationary random fields. Theorems 1.4 and 1.5 below will give some results on the question of what "mixing rates" can occur for such random fields that are also N-tuplewise independent.

A couple of background facts are in order. First (even without stationarity), for the case $d = 1$ (random sequences) one trivially has that $\rho^*(n) = \rho'(n)$. Second, if

the random field X is strictly stationary, then (i) for the case $d \geq 2$, one has that $4\alpha(n) \leq \rho(n) \leq 2\pi\alpha(n)$, and (ii) for any positive integer d (including $d = 1$), analogous inequalities hold (in comparison to $\rho'(n)$ and $\rho^*(n)$ respectively) for dependence coefficients $\alpha'(n)$ and $\alpha^*(n)$ that are defined à la (1.6) and (1.7) using the measure of dependence $\alpha(\mathcal{A}, \mathcal{B})$. (See [4, v3, Theorem 29.12].) Thus there is no real reason to formally define such dependence coefficients $\alpha'(n)$ and $\alpha^*(n)$; and the treatment of the dependence coefficient $\alpha(n)$ (separately from $\rho(n)$) is meaningful only in the case $d = 1$.

We shall now state the two main results of this note, and then give some further motivations for them.

Theorem 1.4. *Suppose d and N are positive integers such that $N \geq 2$. Suppose $(c_1, c_2, c_3, \ldots)$ is a nonincreasing sequence of numbers in the closed unit interval $[0, 1]$. Then there exists a strictly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$ with the following properties:*
(A) The random variable $X_{\mathbf{0}}$ is uniformly distributed on the interval $[0, 1]$.
(B) The random field X satisfies N-tuplewise independence.
(C) For each $n \in \mathbf{N}$, $4\alpha(n) = \rho(n) = \rho'(n) = \rho^*(n) = c_n$.

Theorem 1.5. *Suppose d and N are integers such that $d \geq 2$ and $N \geq 2$. Suppose $(c_1, c_2, c_3, \ldots)$ is a nonincreasing sequence of numbers in the closed unit interval $[0, 1]$. Then there exists a strictly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$ with the following properties:*
(A) The random variable $X_{\mathbf{0}}$ is uniformly distributed on the interval $[0, 1]$.
(B) The random field X satisfies N-tuplewise independence.
(C) For each $n \in \mathbf{N}$, $\rho^*(n) = 1$; also $\rho(1) = \rho'(1) = 1$.
(D) For each $n \geq 2$, $\rho(n) = \rho'(n) = c_n$.

In these two theorems, there is no assumption that $c_n \to 0$ as $n \to \infty$. For some limit theory for random fields under just the "partial mixing" condition $\lim_{n\to\infty} \rho^*(n) < 1$, see e.g. Peligrad and Gut [21].

The constructions in the proofs of Theorems 1.4 and 1.5 are based on a well known simple class of random vectors (given in Definition 2.5 in Section 2) with very nice properties related to both N-tuplewise independence and maximal correlation coefficients. However, this class does not seem to provide the leverage for (i) "separating" the dependence coefficients $\rho(n)$ and $\rho'(n)$ (as was done in the construction in [5] alluded to above) or (ii) getting a value of $\rho(1)$ (or $\rho'(1)$) that is less than that of $\rho^*(1)$. In Theorem 1.5, the explicit mention of the equations $\rho(1) = \rho'(1) = 1$ is intended only as a reminder of that latter fact.

In [4, v3, Theorem 26.8 and its subsequent Note 3] it was shown that for strictly stationary random sequences (the case $d = 1$) with a purely non-atomic marginal distribution, the simultaneously behavior of the dependence coefficients $\alpha(n)$, $\rho(n)$, and $\rho^*(n)$ could be practically arbitrary subject to (1.8) and the sentence after it. (In that "subse-

quent Note 3," the word "atomic" should be "nonatomic.") In [5, Theorem 1.9], it was shown that for strictly stationary random fields in the case $d \geq 2$ with a purely non-atomic marginal distribution, an analogous fact holds for $\rho(n)$, $\rho'(n)$, and $\rho^*(n)$. However, those two constructions did not address the question of whether such examples exist in which the bivariate distributions could have nice "extra" properties such as the ones assumed in the result of Tone [24] cited above. To a certain extent, Theorems 1.4 and 1.5 address that question. Theorem 1.4 shows that for general $d \geq 1$, the cited result in [24], which involves ρ'-mixing with a mixing rate that can be arbitrarily slow along with the "extra" assumptions alluded to above on the marginal and bivariate distributions, cannot be obtained from any similar result involving ρ-mixing with a "mixing-rate" assumption (such as the "logarithmic" mixing-rate assumption $\sum_{n=1}^{\infty} \rho(2^n) < \infty$ that plays a key role in much of the limit theory under ρ-mixing). For $d = 1$ it also shows that any similar result (that includes those "extra" assumptions) under ρ-mixing (or even ρ^*-mixing) with a particular mixing-rate assumption, cannot be obtained from an analogous result involving α-mixing with a faster mixing rate. Theorem 1.5 shows that for the case $d \geq 2$, the cited result in [24] also cannot be obtained from any similar result under ρ^*-mixing or even under assumptions that include $\rho^*(n) < 1$ for some $n \geq 1$.

Theorem 1.4 will be proved in Section 3 below, after some preliminary work in Section 2. Then Theorem 1.5 will be proved in Section 4 (with Theorem 1.4 itself serving as a key "building block" in that proof).

2. Preliminaries

First here are some notations that will be used in the rest of this paper.

For a given pair of nonempty sets A and B, the notation B^A will of course mean the set of all mappings from A to B. A notation such as $(r_s, s \in S)$ will refer to a particular mapping from a given nonempty set S to $\mathbf{R}$ (the one in which $s \mapsto r_s$ for each $s \in S$).

When a notation of the form a^b appears in a subscript, superscript, or exponent, it will be written $a \uparrow b$.

For any given set $S \in \mathbf{Z}^d$ and any $v \in \mathbf{Z}^d$, the notation $S + v$ will of course refer to the set of elements of the form $s + v$ for $s \in S$.

The cardinality of a given set S will be denoted $\operatorname{card} S$.

Now in order to avoid some clutter in arguments later on, we shall give a review of three standard techniques for constructing strictly stationary random fields from other (e.g. "building block") random fields.

Remark 2.1. (A) Suppose d is a positive integer. Suppose that for each $n \in \mathbf{N}$, $Y^{(n)} := (Y_k^{(n)}, k \in \mathbf{Z}^d)$ is a strictly stationary random field. Suppose these random fields $Y^{(n)}, n \in \mathbf{N}$ are independent of each other. Suppose $f : \mathbf{R}^{\mathbf{N}} \to \mathbf{R}$ is a Borel function, and the random field $X := (X_k, k \in \mathbf{Z}^d)$ is defined by $X_k := f(Y_k^{(1)}, Y_k^{(2)}, Y_k^{(3)}, \ldots)$ for $k \in \mathbf{Z}^d$. Then the random field X is strictly stationary.

(B) In the context of (A) above, if also $N \geq 2$ is an integer and each of the random

fields $Y^{(n)}, n \in \mathbf{N}$ satisfies N-tuplewise independence, then the random field X satisfies N-tuplewise independence.

Remark 2.1(A) gives a standard scheme that was apparently first used by Doeblin [11] for the case $d = 1$ (random sequences), in his construction of an i.i.d. sequence whose marginal distribution is "universal," i.e. in the domain of partial attraction to all infinitely divisible laws. To verify statement (A), one can first observe with a simple argument that for any Borel sets $B_{n,k} \subset \mathbf{R}$, $n \in \mathbf{N}$, $k \in \mathbf{Z}^d$, and any $i \in \mathbf{Z}^d$,

$$P\Big(\bigcap_{(n,k)\in\mathbf{N}\times(\mathbf{Z}\uparrow d)} \{Y_k^{(n)} \in B_{n,k}\}\Big) = P\Big(\bigcap_{(n,k)\in\mathbf{N}\times(\mathbf{Z}\uparrow d)} \{Y_{k+i}^{(n)} \in B_{n,k}\}\Big).$$

After that, statement (A) (that is, the strict stationarity of X) follows from an elementary measure-theoretic argument. Then Property (B) follows by an elementary argument.

Remark 2.2. (A) Suppose d is a positive integer. Suppose $Y := (Y_k, k \in \mathbf{Z}^d)$ is a (not necessarily stationary) random field. Suppose that for each $u \in \mathbf{Z}^d$, $Y^{(u)} := (Y_j^{(u)}, j \in \mathbf{Z}^d)$ is a random field whose distribution on (the Borel σ-field on) $\mathbf{R}^{\mathbf{Z}\uparrow d}$ is the same as that of Y. Suppose these random fields $Y^{(u)}$, $u \in \mathbf{Z}^d$ are independent of each other. Suppose $f : \mathbf{R}^{\mathbf{Z}\uparrow d} \to \mathbf{R}$ is a Borel function. Suppose $X := (X_k, k \in \mathbf{Z}^d)$ is the random field defined by $X_k := f((Y_j^{(k-j)}, j \in \mathbf{Z}^d))$ for $k \in \mathbf{Z}^d$. Then the random field X is strictly stationary.

(B) In the context of (A) above, if also $N \ge 2$ is an integer and the random field Y satisfies N-tuplewise independence, then the random field X satisfies N-tuplewise independence.

The scheme in Remark 2.2(A) was employed by Olshen [20] in the case $d = 1$ (random sequences) to "convert" a non-stationary random sequence with certain "ergodic-theoretic" properties (trivial "past" and "future" tail σ-fields and a nontrivial "double" tail σ-field) into a strictly stationary sequence with those properties. For $d \ge 2$, it was similarly employed in [2] to construct a strictly stationary random field $X := (X_k, k \in \mathbf{Z}^d)$ that satisfies $\rho(2) = 0$ and whose "tail σ-field" is (modulo sets of probability 0) identical to $\sigma(X)$ itself. To verify statement (A), one can first observe with a simple argument (slightly cumbersome to write out in more detail) that for any Borel sets $B_{k,j} \subset \mathbf{R}$, $k \in \mathbf{Z}^d$, $j \in \mathbf{Z}^d$, and any $i \in \mathbf{Z}^d$,

$$P\Big(\bigcap_{(k,j)\in(\mathbf{Z}\uparrow d)\times(\mathbf{Z}\uparrow d)} \{Y_j^{(k-j)} \in B_{k,j}\}\Big) = P\Big(\bigcap_{(k,j)\in(\mathbf{Z}\uparrow d)\times(\mathbf{Z}\uparrow d)} \{Y_j^{((k+i)-j)} \in B_{k,j}\}\Big).$$

After that, statement (A) (the strict stationarity of X) follows from an elementary measure-theoretic argument. Then Property (B) follows by an elementary argument.

Remark 2.3. (A) Suppose d is a positive integer and $X := (X_k, k \in \mathbf{Z}^d)$ is a strictly stationary random field. Let $F : \mathbf{R} \to [0, 1]$ denote the (marginal) distribution function of the random variable $X_{\mathbf{0}}$ — that is, $F(x) := P(X_{\mathbf{0}} \le x)$. Let $G : \mathbf{R} \to [0, 1]$ be the function

defined by $G(x) := \lim_{y\to x-} F(y)$. Let $F^{-1} : (0,1) \to \mathbf{R}$ denote the "generalized inverse" function of F — that is, $F^{-1}(u) := \inf\{x \in \mathbf{R} : F(x) \geq u\}$. Let $V := (V_k, k \in \mathbf{Z}^d)$ be a random field of independent, identically distributed random variables, each uniformly distributed on the interval $[0,1]$, with this random field V being independent of the random field X. Define the random field $U := (U_k, k \in \mathbf{Z}^d)$ as follows: For each $k \in \mathbf{Z}^d$,

$$U_k := G(X_k) + V_k \cdot [F(X_k) - G(X_k)]. \tag{2.1}$$

Then this random field U has the following properties: (1) The random field U is strictly stationary. (2) The random variable $U_{\mathbf{0}}$ is uniformly distributed on the interval $[0,1]$. (3) For each $k \in \mathbf{Z}^d$, $X_k = F^{-1}(U_k)$ a.s. (4) For any two nonempty disjoint subsets S and T of $\mathbf{Z}^d$,

$$\alpha(\sigma(U_k, k \in S), \sigma(U_k, k \in T)) = \alpha(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) \quad \text{and} \tag{2.2}$$

$$\rho(\sigma(U_k, k \in S), \sigma(U_k, k \in T)) = \rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)); \tag{2.3}$$

and hence for each positive integer n, (i) $\alpha(U,n) = \alpha(X,n)$, (ii) $\rho(U,n) = \rho(X,n)$, (iii) $\rho'(U,n) = \rho'(X,n)$, and (iv) $\rho^*(U,n) = \rho^*(X,n)$.

(B) In the context of (A) above, if also $N \geq 2$ is an integer and the random field X satisfies N-tuplewise independence, then the random field U satisfies N-tuplewise independence.

The construction in (2.1) is classic. Mixing properties of this construction (à la the entire sentence (4) above) essentially go back to Donald W.K. Andrews and Manfred Denker, independently of each other, in unpublished manuscripts around the year 1982, both in the context of random sequences (the case $d = 1$). (See e.g. the abstract in [10].) One reference for the proof in the case $d = 1$ is [4, v1, Theorem 6.8]. The proof for general $d \geq 1$ is the same as for $d = 1$. (In particular, for (2.2) and (2.3), refer to (2.1) and see [4, v1, Theorem 6.2(I)(II) and Remark 6.3 (its third paragraph)].) In Tone's proof of her result in [24] cited above, part (A) was applied in the context of random fields (general $d \geq 1$), with $X_{\mathbf{0}}$ having an absolutely continuous distribution. Part (B) is a trivial addendum, an immediate consequence of (2.1) and the properties of the random field V.

In our application of the schemes above, the following lemma will play a key role. It is due to Csáki and Fischer [8, Theorem 6.2]. For a generously detained presentation of its proof, see e.g. [4, v1, Theorem 6.1].

Lemma 2.4. *Suppose $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots$ and $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \ldots$ are σ-fields, and the σ-fields $\mathcal{A}_n \vee \mathcal{B}_n$, $n \in \mathbf{N}$ are independent. Then*

$$\rho\Big(\bigvee_{n\in\mathbf{N}} \mathcal{A}_n, \bigvee_{n\in\mathbf{N}} \mathcal{B}_n\Big) = \sup_{n\in\mathbf{N}} \rho(\mathcal{A}_n, \mathcal{B}_n).$$

The next definition (an old classic one), and the lemma right after it, will play a key role in obtaining N-tuplewise independence in the constructions for Theorems 1.4 and 1.5.

The redundancy in this definition may help highlight its main features.

Definition 2.5. For any integer $m \geq 3$, define the probability measures $\nu_0^{(m)}$ and $\nu_1^{(m)}$ on $\{-1,1\}^m$ (the set of all m-tuples of $+1$'s and -1's) as follows: For each $x := (x_1, x_2, \dots, x_m) \in \{-1,1\}^m$,

$$\nu_0^{(m)}(x) := 1/2^m \quad \text{and} \quad \nu_1^{(m)}(x) := \begin{cases} 1/2^{m-1} & \text{if } x_1 \cdot x_2 \cdot \ldots \cdot x_m = 1 \\ 0 & \text{if } x_1 \cdot x_2 \cdot \ldots \cdot x_m = -1. \end{cases}$$

For any integer $m \geq 3$ and any $\theta \in (0,1)$, define the probability measure $\nu_\theta^{(m)}$ on $\{-1,1\}^m$ by

$$\nu_\theta^{(m)} := (1-\theta)\nu_0^{(m)} + \theta\nu_1^{(m)}. \tag{2.4}$$

By simple arithmetic, for a given $x := (x_1, x_2, \dots, x_m) \in \{-1,1\}^m$, the number $\nu_\theta^{(m)}(x)$ is equal to $(1+\theta)/2^m$ (resp. $(1-\theta)/2^m$) if $x_1 \cdot x_2 \cdot \ldots \cdot x_m = 1$ (resp. if $x_1 \cdot x_2 \cdot \ldots \cdot x_m = -1$).

Lemma 2.6. *Suppose m is an integer such that $m \geq 3$, and $\theta \in [0,1]$. Suppose $V := (V_1, V_2, \dots, V_m)$ is an $\{-1,1\}^m$-valued random vector whose distribution is $\nu_\theta^{(m)}$. Then the following statements hold:*

(A) For any permutation σ of $\{1,2,\dots,m\}$, the distribution of the random vector $V^{(\sigma)} := (V_{\sigma(1)}, V_{\sigma(2)}, \dots, V_{\sigma(m)})$ is $\nu_\theta^{(m)}$ as well.

(B) For each $i \in \{1,2,\dots,m\}$, one has that $P(V_i = 1) = P(V_i = -1) = 1/2$.

(C) Every $m-1$ of the random variables $V_1, V_2, \dots, V_m$ are independent.

(D) If S and T are nonempty disjoint sets whose union is $\{1,2,\dots,m\}$, then

$$4\alpha(\sigma(V_i, i \in S), \sigma(V_i, i \in T)) = \rho(\sigma(V_i, i \in S), \sigma(V_i, i \in T)) = \theta.$$

Property (A) in Lemma 2.6 is an immediate consequence of the sentence after (2.4). Properties (B), (C), and (D) are elementary and were shown in [1, Lemma 3.2]. Because of property (A), Definition 2.5 can be put in a more flexible form, as follows:

Definition 2.7. For any finite set S with $m := \operatorname{card} S \geq 3$ and any $\theta \in [0,1]$, a collection $(V_i, i \in S)$ of $\{-1,1\}$-valued random variables is said to have the distribution $\nu_\theta^{(m)}$ if the distribution of the random vector $V := (V_{s(1)}, V_{s(2)}, \dots, V_{s(m)})$ is $\nu_\theta^{(m)}$ for any, hence every, ordered listing $s(1), s(2), \dots, s(m)$ of the elements (each exactly once) of S.

3. Proof of Theorem 1.4

The proof of Theorem 1.4 will start with the following lemma:

Lemma 3.1. *Suppose d, N, and n are positive integers such that $N \geq 2$. Suppose $\theta \in [0,1]$. Then there exists a strictly stationary, N-tuplewise independent random field $X := (X_k, k \in \mathbf{Z}^d)$ such that (i) $4\alpha(X,n) = \rho^*(X,1) = \theta$ and (ii) $\rho^*(X,n+1) = 0$.*

Note that by (1.8), properties (i) and (ii) here imply that
(a) $\rho'(X,1) = \rho(X,1) = 4\alpha(X,1) = \rho^*(X,n) = \rho'(X,n) = \rho(X,n) = \theta$ and
(b) $\rho'(X,n+1) = \rho(X,n+1) = \alpha(X,n+1) = 0$.
Such consequences of (1.8) will occur regularly in the rest of this note.

Proof. The proof will be divided into two "steps."

Step 1. Let $M \geq 2$ be an integer such that $M^d - 1 \geq N$. Define the set $\Lambda := \{0, n, 2n, \dots, (M-1)n\}^d$. This set has cardinality M^d.

Let $Y := (Y_k, k \in \mathbf{Z}^d)$ be a random field with the following properties: (i) The random variables $Y_k, k \in \Lambda$ take only the values -1 and $+1$; and the distribution of this collection $(Y_k, k \in \Lambda)$ is $\nu_\theta^{(M\uparrow d)}$ (see Definition 2.7). (ii) The random variables $Y_k, k \in \mathbf{Z}^d - \Lambda$ are constant, defined by $Y_k := 0$.

By Lemma 2.6(C), every $M^d - 1$ of the random variables $Y_k, k \in \Lambda$ are independent. It follows trivially that the entire random field Y satisfies $(M^d - 1)$-tuplewise independence, and hence (by the first sentence of Step 1 here) N-tuplewise independence.

By Lemma 2.6(D), if S and T are nonempty, disjoint subsets of Λ and their union is Λ, then

$$4\alpha(\sigma(Y_k, k \in S), \sigma(Y_k, k \in T)) = \rho(\sigma(Y_k, k \in S), \sigma(Y_k, k \in T)) = \theta.$$

It follows from a simple argument that

$$4\alpha(Y,n) = \rho^*(Y,1) = \theta. \tag{3.1}$$

(To see that $4\alpha(Y,n) = \theta$, refer to (1.4) and consider the index sets $\{k \in \mathbf{Z}^d : k_1 \leq 0\}$ and $\{k \in \mathbf{Z}^d : k_1 \geq n\}$; each element of Λ belongs to one of those two sets, and each of those two sets has at least one element of Λ.)

Also, if S and T are nonempty disjoint subsets of $\mathbf{Z}^d$ such that $\text{dist}(S,T) \geq n+1$, and S and T each have at least one element of Λ, then some element of Λ is in neither S nor T. (If $j \in \Lambda \cap S$ and $k \in \Lambda \cap T$, then there exists a "chain" ($j := \kappa(0), \kappa(1), \kappa(2), \dots, \kappa(p) := k$) of elements of Λ such that for each $i \in \{1, 2, \dots, p\}$, $\|\kappa(i) - \kappa(i-1)\| = n$ — with $\kappa(i)_u - \kappa(i-1)_u = \pm n$ for some $u \in \{1, \dots, d\}$; and if also $\Lambda \subset S \cup T$, then for some $i \in \{1, 2, \dots, p\}$, one has $\kappa(i-1) \in S$ and $\kappa(i) \in T$, forcing $\text{dist}(S,T) \leq n$, a contradiction.) It follows from Lemma 2.6(C) that

$$\rho^*(Y, n+1) = 0. \tag{3.2}$$

Step 2. For each $u \in \mathbf{Z}^d$, let $Y^{(u)} := (Y_j^{(u)}, j \in \mathbf{Z}^d)$ be a random field with the same distribution (on $\{-1,0,1\}^{\mathbf{Z}\uparrow d}$) as the random field Y above. Let these random fields $Y^{(u)}$, $u \in \mathbf{Z}^d$ be constructed in such a way that they are independent of each other. For

technical convenience, assume that for *every* $\omega \in \Omega$ and every $u \in \mathbf{Z}^d$, $Y_j^{(u)}(\omega) \in \{-1, 1\}$ (resp. $= 0$) if $j \in \Lambda$ (resp. $j \in \mathbf{Z}^d - \Lambda$).

Referring to the first paragraph of Step 1, let ϕ be a one-to-one function from Λ onto $\{0, 1, \dots, M^d - 1\}$. Define the random field $X := (X_k, k \in \mathbf{Z}^d)$ as follows: For each $k \in \mathbf{Z}^d$,

$$X_k := \sum_{j \in \Lambda} 2^{\phi(j)} \cdot [(Y_j^{(k-j)} + 1)/2]. \tag{3.3}$$

By Remark 2.2(A), this random field X is strictly stationary; and by Remark 2.2(B) and the third paragraph of Step 1 above, it satisfies N-tuplewise independence. Our remaining task is to verify properties (i) and (ii) in the statement of Lemma 3.1.

The quantity $(1 + a)/2$ equals 0 (resp. 1) if $a = -1$ (resp. 1). Hence by the first paragraph of Step 2 here, for a given $k \in \mathbf{Z}^d$, the random variables $(Y_j^{(k-j)} + 1)/2$, $j \in \Lambda$ each take just the values 0 and 1. Also, the binary expansion of a given positive integer is unique. It follows easily from (3.3) that for any given $k \in \mathbf{Z}^d$,

$$\sigma(X_k) = \sigma(Y_j^{(k-j)}, j \in \Lambda) = \sigma(Y_j^{(k-j)}, j \in \mathbf{Z}^d). \tag{3.4}$$

Here the second equality follows from the fact that the random variables $Y_j^{(k-j)}$, $j \in \mathbf{Z}^d - \Lambda$ are constant (in fact 0).

Now suppose S and T are nonempty subsets of $\mathbf{Z}^d$. By (3.4),

$$\sigma(X_k, k \in S) = \bigvee_{j \in \mathbf{Z} \uparrow d} \sigma(Y_\ell^{(j)}, \ell \in S - j), \tag{3.5}$$

and the same holds with each S replaced by T. It follows trivially that

$$\rho\Big(\sigma(X_k, k \in S), \sigma(X_k, k \in T)\Big) \ge \rho\Big(\sigma(Y_k^{(\mathbf{0})}, k \in S), \sigma(Y_k^{(\mathbf{0})}, k \in T)\Big), \tag{3.6}$$

and that the same holds with the symbol ρ replaced on both sides by α. Also, by (3.5) and its analog for T, and Lemma 2.4, one has that

$$\rho\Big(\sigma(X_k, k \in S), \sigma(X_k, k \in T)\Big) = \sup_{j \in \mathbf{Z} \uparrow d} \rho\Big(\sigma(Y_\ell^{(j)}, \ell \in S - j), \sigma(Y_\ell^{(j)}, \ell \in T - j)\Big). \tag{3.7}$$

If the sets S and T are disjoint, then for any given $j \in \mathbf{Z}^d$, the sets $S - j$ and $T - j$ are disjoint. Hence by (3.1) and (3.7), $\rho^*(X, 1) \le \theta$. However, by (3.6) and (3.1), one also has that $\rho^*(X, 1) \ge \rho^*(Y^{(\mathbf{0})}, 1) = \theta$. Hence

$$\rho^*(X, 1) = \theta. \tag{3.8}$$

Now if the sets S and T satisfy $\text{dist}(S, T) \ge n+1$, then for each $j \in \mathbf{Z}^d$, $\text{dist}(S-j, T-j) \ge n + 1$, and hence by (3.7) and (3.2), $\rho(\sigma(X_k, k \in S), \sigma(X_k, k \in T)) = 0$. It follows

that $\rho^*(X, n+1) = 0$. Also, by (3.1) and the analog of (3.6) for $\alpha(\dots)$, one has that $4\alpha(X, n) \geq 4\alpha(Y^{(\mathbf{0})}, n) = \theta$. Hence by (1.8) and (3.8), $4\alpha(X, n) = \theta$. All equations in properties (i) and (ii) in Lemma 3.1 have been verified, and the proof of Lemma 3.1 is complete.

Proof of Theorem 1.4. Suppose d and N and the sequence $(c_1, c_2, c_3, \dots)$ are as in the statement of Theorem 1.4. For each positive integer n, applying Lemma 3.1, let $Z^{(n)} := (Z_k^{(n)}, k \in \mathbf{Z}^d)$ be a strictly stationary, N-tuplewise independent random field such that

$$4\alpha(Z^{(n)}, n) = \rho^*(Z^{(n)}, 1) = c_n \quad \text{and} \quad \rho^*(Z^{(n)}, n+1) = 0. \tag{3.9}$$

Let these random fields $Z^{(n)}, n \in \mathbf{N}$ be constructed in such a way that they are independent of each other.

Let $\psi : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \dots \to \mathbf{R}$ be a function which is one-to-one, onto, and bimeasurable (with respect to the Borel σ-fields). (Such functions are well known to exist.) Define the random field $X := (X_k, k \in \mathbf{Z})$ as follows: For each $k \in \mathbf{Z}^d$,

$$X_k := \psi(Z_k^{(1)}, Z_k^{(2)}, Z_k^{(3)}, \dots). \tag{3.10}$$

By Remark 2.1(A)(B), the random field X is strictly stationary and satisfies N-tuplewise independence. By (3.10) and the properties of the function ψ, one has that for each element $k \in \mathbf{Z}^d$, $\sigma(X_k) = \bigvee_{n \in \mathbf{N}} \sigma(Z_k^{(n)})$. Hence for each $n \in \mathbf{N}$, by (3.9),

$$\alpha(X, n) \geq \alpha(Z^{(n)}, n) = c_n/4. \tag{3.11}$$

It also follows from Lemma 2.4 that for each $n \in \mathbf{N}$

$$\rho^*(X, n) = \sup_{m \in \mathbf{N}} \rho^*(Z^{(m)}, n). \tag{3.12}$$

Now for a given $m \in \mathbf{N}$ and $n \in \mathbf{N}$, one has that $\rho^*(Z^{(m)}, n) = 0$ by (3.9) if ($n \geq 2$ and) $m < n$; and if instead $m \geq n$ then $\rho^*(Z^{(m)}, n) = c_m \leq c_n$ by (3.9), (1.8), and the hypothesis of Theorem 1.4. Hence for each $n \in \mathbf{N}$, $\rho^*(X, n) = c_n$ by (3.12). Hence for each $n \in \mathbf{N}$, by (3.11) and (1.8), the equalities in property (C) of Theorem 1.4 all hold for the random field X here.

All that remains is to convert the random field X into one in which the (marginal) distribution of $X_{\mathbf{0}}$ is uniform on $[0, 1]$ without changing the other properties stated in Theorem 1.4. On accomplishes that by applying Remark 2.3(A)(B). That completes the proof of Theorem 1.4.

4. Proof of Theorem 1.5

The proof of Theorem 1.5 will first involve two lemmas. The constructions in the proofs of those lemmas will be somewhat related to the construction in [2].

Lemma 4.1. *Suppose d, N, and n are positive integers such that $d \geq 2$ and $N \geq 2$. Then there exists a strictly stationary, N-tuplewise independent random field $X := (X_k, k \in \mathbf{Z}^d)$ such that (i) $\rho^*(X, n) = 1$ and (ii) $\rho(X, 1) = 1$ and $\rho(X, 2) = 0$.*

Proof. The proof will be divided into two "steps."

Step 1. Increasing n if necessary, we assume without loss of generality that

$$n > N. \tag{4.1}$$

Let Γ_0 denote the set of all points $k := (k_1, k_2, \ldots, k_d) \in \{-n, -n+1, -n+2, \ldots, n\}^d$ such that $k_u \in \{-n, n\}$ for at least one index $u \in \{1, 2, \ldots, d\}$. That is, Γ_0 is the "boundary" or "shell" of the "cube" $\{-n, -n+1, \ldots, n\}^d$. Define the set $\Gamma := \Gamma_0 \cup \{\mathbf{0}\}$. Note that

$$\mathrm{dist}\Big(\Gamma_0, \{\mathbf{0}\}\Big) = n. \tag{4.2}$$

Also, the sets $\{k \in \mathbf{Z}^d : k_1 \leq 0\}$ and $\{k \in \mathbf{Z}^d : k_1 \geq 1\}$ each contain elements of Γ. Also, for a given $j \in \mathbf{Z}$ and a given index $u \in \{1, 2, \ldots, d\}$, one of the following three statements holds (depending on whether $j \geq n+1$, $j \leq -n-1$, or $-n \leq j \leq n$): either (a) $\Gamma \subset \{k \in \mathbf{Z}^d : k_u \leq j-1\}$, or (b) $\Gamma \subset \{k \in \mathbf{Z}^d : k_u \geq j+1\}$, or (c) the ("slice") set $\{k \in \mathbf{Z}^d : k_u = j\}$ contains elements of Γ (here the assumption $d \geq 2$ is used). These trivial observations will be useful shortly.

Let $Y := (Y_k, k \in \mathbf{Z}^d)$ be a random field with the following properties: (i) The random variables $Y_k, k \in \Gamma$ take only the values -1 and $+1$; and the distribution of this collection $(Y_k, k \in \Gamma)$ is $\nu_1^{(\mathrm{card}\,\Gamma)}$ (see Definition 2.7 and note the "extreme" value 1 in the subscript here). (ii) The random variables $Y_k, k \in \mathbf{Z}^d - \Gamma$ are constant, defined by $Y_k := 0$. One has that

$$\rho^*(Y, n) = 1 \quad \text{and} \quad \rho(Y, 1) = 1 \quad \text{and} \quad \rho(Y, 2) = 0. \tag{4.3}$$

In (4.3), the first equality holds by (4.2) and Lemma 2.6(D), and the second equality holds by Lemma 2.6(D) and the sentence right after (4.2). The third equality in (4.3) holds by the second sentence after (4.2). (In the definition of $\rho(Y, 2)$, one can represent the pairs of index sets à la (a) and (b) of the second sentence after (4.2), and for u, j such that (c) in that sentence holds, one applies Lemma 2.6(C)).

By Lemma 2.6(C), every $(\mathrm{card}\,\Gamma) - 1$ of the random variables $Y_k, k \in \Gamma$ are independent. Now trivially $\mathrm{card}\,\Gamma > n$; and now one has by (4.1) that every N of the random variables $Y_k, k \in \Gamma$ are independent. It follows trivially that the entire random field Y satisfies N-tuplewise independence.

Step 2. Now we follow the argument in Step 2 of the proof of Lemma 3.1 in Section 3. Here we shall just describe the changes.

In the first paragraph of Step 2 there, replace each Λ by Γ.

In the second paragraph of Step 2 there, the changes are as follows: (a) One lets ϕ be a one-to-one function from Γ onto $\{0, 1, \ldots, (\mathrm{card}\,\Gamma) - 1\}$. (b) In eq. (3.3), the symbol

Λ is replaced by Γ. (c) The "remaining task" is to verify (i) and (ii) in the statement of Lemma 4.1 (instead of in Lemma 3.1).

In the third paragraph of Step 2 (including in eq. (3.4)), the symbol Λ is replaced throughout by Γ.

The fourth paragraph of Step 2 (with eqs. (3.5), (3.6), and (3.7)) remains unchanged. The final (i.e. fifth) paragraph of Step 2 is modified as follows: First, one simply uses (4.3) and (3.6) (and (1.8)) to obtain that $\rho^*(X, n) = 1$ and $\rho(X, 1) = 1$. Then one observes that if S and T are nonempty subsets of $\mathbf{Z}^d$ that are on opposite sides of some "slice" (a set of the form $\{k \in \mathbf{Z}^d : k_u = h\}$ where $h \in \mathbf{Z}^d$ and $u \in \{1, \ldots, d\}$), then for any $j \in \mathbf{Z}^d$, that is true as well for the sets $S - j$ and $T - j$ (with respect to a possibly different "slice"). From that and (3.7) and the third equality in (4.3), one obtains $\rho(X, 2) = 0$. Thus properties (i) and (ii) in Lemma 4.1 have been verified, and the proof of that lemma is complete.

Lemma 4.2. *Suppose d and N are integers such that $d \geq 2$ and $N \geq 2$. Then there exists a strictly stationary, N-tuplewise independent random field $X := (X_k, k \in \mathbf{Z}^d)$ such that (i) $\rho^*(X, n) = 1$ for all $n \in \mathbf{N}$ and (ii) $\rho(X, 1) = 1$ and $\rho(X, 2) = \rho'(X, 2) = 0$.*

Proof. One follows the Proof of Theorem 1.4 (after the proof Lemma 3.1) in Section 3, but with the following changes:

In the first paragraph of that proof, the mention of a sequence $(c_1, c_2, c_3, \ldots)$ is to be omitted, one applies Lemma 4.1 (instead of Lemma 3.1), and eq. (3.9) is replaced by

$$\rho^*(Z^{(n)}, n) = 1 \quad \text{and} \quad \rho(Z^{(n)}, 1) = 1 \quad \text{and} \quad \rho(Z^{(n)}, 2) = 0. \tag{4.4}$$

Eq. (3.10) and its entire paragraph, and also the first two sentences after (3.10), are left unchanged. From (3.10) (which yields $\rho^*(X, n) \geq \rho^*(Z^{(n)}, n)$ for each $n \in \mathbf{N}$) and (4.4), one obtains that $\rho^*(X, n) = 1$ for all $n \in \mathbf{N}$. Similarly from (3.10) and (4.4) one trivially obtains $\rho(X, 1) = 1$; and by (3.10), (4.4), and (say) Lemma 2.4, one also obtains $\rho(X, 2) = 0$. From that last equality, one also has that $\rho'(X, 2) = 0$, by an elementary argument (see e.g. [4, v3, Proposition 29.5]). That completes the proof of Lemma 4.2.

Proof of Theorem 1.5. Suppose the integers d and N and the sequence $(c_1, c_2, c_3, \ldots)$ are as in the statement of Theorem 1.5. Let $Y := (Y_k, k \in \mathbf{Z}^d)$ be a random field that satisfies all properties stated for the random field X in Theorem 1.4. Let $Z := (Z_k, k \in \mathbf{Z}^d)$ be a random field that satisfies all properties stated for the random field X in Lemma 4.2. Let these two random fields Y and Z be constructed in such as way that they are independent of each other. Let $\zeta : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a function which is one-to-one, onto, and bimeasurable (with respect to the Borel σ-fields). Define the random field $X := (X_k, k \in \mathbf{Z}^d)$ as follows: For each $k \in \mathbf{Z}^d$,

$$X_k := \zeta(Y_k, Z_k). \tag{4.5}$$

By Remark 2.1(A)(B) (applied to Y and Z and, say, a sequence of degenerate random fields whose random variables take only the value 0), the random field X is strictly

stationary and N-tuplewise independent. By (4.5) and the properties of the function ζ, one has that for any given $k \in \mathbf{Z}^d$,

$$\sigma(X_k) = \sigma(Y_k, Z_k). \tag{4.6}$$

From (4.6) and Lemma 4.2 (for the random field Z), one immediately obtains that $\rho^*(X, n) = 1$ for all $n \in \mathbf{N}$, and that $\rho(X, 1) = 1$, and hence also $\rho'(X, 1) = 1$. Also, for each $n \geq 2$, one has by Lemma 2.4, followed by the properties in Theorem 1.4 (for Y) and Lemma 4.2 (for Z),

$$\rho(X, n) = \max\{\rho(Y, n), \rho(Z, n)\} = \max\{c_n, 0\} = c_n,$$

and the same holds with each symbol ρ replaced by ρ'.

Now all that remains is to convert the random field X into one in which the (marginal) distribution of $X_{\mathbf{0}}$ is uniform on $[0, 1]$ without changing the other properties stated in Theorem 1.5. One accomplishes that with an application of Remark 2.3(A)(B). That completes the proof of Theorem 1.5.

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